VARIATIONAL THEOREM OF GENERALIZED

COSSERAT-MEDIUM THERMOELASTICITY

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An equation which represents the content of the variational theorem of generalized Cosserat-medium elasticity is introduced. It is shown that, on the basis of this theorem, the basic energy equation of this medium may be obtained.

In investigating such rapidly occurring processes as the heating of a solid by a laser pulse, it is necessary to take the finite rate of heat propagation into account, which may be done by introducing an additional term in the Fourier heat-conduction law [1, 2]. Heat-conduction and thermoelasticity theories based on this hypothesis are spoken of as generalized theories. The basic equations and theorems of generalized thermoelasticity for a symmetric medium are given in [3]. On the other hand, as noted in [4], phenomena occurring in granular and polycrystalline media are better described by the Cosserat model than by the classical symmetric-medium model. Cosserat-medium equations of equilibrium and motion are given in [4, 5]. The Cosserat-medium thermoelasticity equations, for an infinite rate of heat propagation, were obtained and analyzed in [4, 6]. The complete system of generalized-thermoelasticity equations for a Cosserat medium was obtained in [7].

The Cosserat-medium equations of motion given in [4-8] take the form

$$\nabla\cdot\mathbf{T}+\mathbf{X}=\rho\ddot{\mathbf{u}},$$
$$\nabla\cdot\mathbf{M}+\mathbf{Y}-2\mathbf{a}^{T}=\mathbf{I}\cdot\ddot{\boldsymbol{\omega}},\tag{1}$$

where $\mathbf{u}$ is a small-displacement vector; $\mathbf{a}^{\mathrm{T}}$ is a vector corresponding to the tensor $\mathbf{T}$ [9]. The defining equations, given in [4], take the form

$$\mathbf{T}=2\mu\boldsymbol{\gamma}^{+}+2\alpha\boldsymbol{\gamma}^{-}+(\lambda\boldsymbol{\gamma}\cdot\cdot\mathbf{E}-\nu\Theta_0\vartheta)\,\mathbf{E};$$
$$\mathbf{M}=2\gamma\boldsymbol{\varkappa}^{+}+2\varepsilon\boldsymbol{\varkappa}^{-}+\beta(\boldsymbol{\varkappa}\cdot\cdot\mathbf{E})\,\mathbf{E};\tag{2}$$
$$s=\nu\boldsymbol{\gamma}\cdot\cdot\mathbf{E}+m\Theta_0\vartheta,$$

where

$$\boldsymbol{\varkappa}=\nabla\boldsymbol{\omega};\quad \boldsymbol{\gamma}=\nabla\mathbf{u}+\boldsymbol{\omega}\times\mathbf{E}.\tag{3}$$

Here $\vartheta=(\Theta-\Theta_0)\Theta_0^{-1}$ is the relative temperature deviation from the initial value; μ, α, λ, ν, γ, ε, β, m are constants characterizing the elastic and thermophysical properties of the medium, introduced in [4]; the superscripts plus and minus denote the symmetric and antisymmetric tensor components, respectively.

Consider a body of volume V bounded by a surface Φ, at which bulk forces $\mathbf{X}$, bulk moments $\mathbf{Y}$, surface forces $\mathbf{F}$, and surface moments $\mathbf{P}$ act. It is assumed that heat liberation with a bulk density w occurs inside the body, and that there is a temperature distribution H at the surface.

Introducing the virtual displacement $\delta\mathbf{u}$ and the virtual rotation $\delta\boldsymbol{\omega}$, scalar multiplication of the first relation in Eq. (1) by $\delta\mathbf{u}$ and the second by $\delta\boldsymbol{\omega}$ is performed; they are then added and integrated over the volume, to given

$$\int\limits_V\{(\nabla\cdot\mathbf{T})\cdot\delta\mathbf{u}+\mathbf{X}\cdot\delta\mathbf{u}-\rho\ddot{\mathbf{u}}\cdot\delta\mathbf{u}+(\nabla\cdot\mathbf{M})\cdot\delta\boldsymbol{\omega}+\mathbf{Y}\cdot\delta\boldsymbol{\omega}-2\mathbf{a}^{T}\cdot\delta\boldsymbol{\omega}-(\mathbf{I}\cdot\ddot{\boldsymbol{\omega}})\cdot\delta\boldsymbol{\omega}\}\,dV=0.\tag{4}$$

Using the Ostrogradskii—Gauss theorem and the first two defining relations in Eq. (2), Eq. (4) is brought to the form

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$$\int_V [(\mathbf{X}-\rho\ddot{\mathbf{u}})\cdot\delta\mathbf{u}+(\mathbf{Y}-\mathbf{I}\cdot\ddot{\boldsymbol{\omega}})\cdot\delta\boldsymbol{\omega}]\,dV+\int_\Phi [\mathbf{F}\cdot\delta\mathbf{u}+\mathbf{P}\cdot\delta\boldsymbol{\omega}]\,d\Phi+\nu\Theta_0\int_V \vartheta(\delta\boldsymbol{\gamma}\cdot\cdot\mathbf{E})\,dV-\delta W=0, \tag{5}$$

where

$$W=\int_V \{\mu(\boldsymbol{\gamma}^+\cdot\cdot\boldsymbol{\gamma}^+)+\alpha(\boldsymbol{\gamma}^-\cdot\cdot\boldsymbol{\gamma}^-)+\gamma(\boldsymbol{\varkappa}^+\cdot\cdot\boldsymbol{\varkappa}^+)+\varepsilon(\boldsymbol{\varkappa}^-\cdot\cdot\boldsymbol{\varkappa}^-)+\frac{\lambda}{2}(\mathbf{E}\cdot\cdot\boldsymbol{\gamma})^2+\frac{\beta}{2}(\mathbf{E}\cdot\cdot\boldsymbol{\varkappa})^2\}\,dV. \tag{6}$$

It is now necessary to obtain an equation analogous to the Biot equation. It may be introduced as in [3]. In contrast to [3], the density of bulk heat liberation is assumed to be nonzero. Consider the entropy-balance equation [10]

$$\Theta_0\dot{s}(1+\vartheta)=-\nabla\cdot\mathbf{q}+w;\quad \Theta_0\dot{s}\approx-\nabla\cdot\mathbf{q}+w \tag{7}$$

and the generalized Fourier law [2]

$$\tau_0\dot{\mathbf{q}}+\mathbf{q}=-k\Theta_0\nabla\vartheta. \tag{8}$$

Introducing the Biot vector $\mathbf{B}$

$$\dot{\mathbf{B}}=\Theta_0^{-1}\mathbf{q}, \tag{9}$$

Eq. (8) is multiplied by the virtual increment $\delta\mathbf{B}$ of the Biot vector and integrated over the volume, to give

$$\int_V \{\tau_0\ddot{\mathbf{B}}+\dot{\mathbf{B}}+k\nabla\vartheta\}\cdot\delta\mathbf{B}\,dV=0. \tag{10}$$

Transforming the result in accordance with the Ostrogradskii–Gauss theorem gives

$$\int_V \{[\tau_0\ddot{\mathbf{B}}+\dot{\mathbf{B}}]\cdot\delta\mathbf{B}-k\vartheta\nabla\cdot\delta\mathbf{B}\}\,dV+k\int_\Phi H\mathbf{n}\cdot\delta\mathbf{B}\,d\Phi=0. \tag{11}$$

Multiplying Eq. (11) by $-\Theta_0 k^{-1}$ and adding Eq. (5) yields a variational equation which represents the content of the variational theorem of generalized Cosserat-medium thermoelasticity

$$\begin{gathered}\int_V [(\mathbf{X}-\rho\ddot{\mathbf{u}})\cdot\delta\mathbf{u}+(\mathbf{Y}-\mathbf{I}\cdot\ddot{\boldsymbol{\omega}})\cdot\delta\boldsymbol{\omega}]\,dV+\int_\Phi [\mathbf{F}\cdot\delta\mathbf{u}+\mathbf{P}\cdot\delta\boldsymbol{\omega}]\,d\Phi\\ +\nu\Theta_0\int_V \vartheta(\delta\boldsymbol{\gamma}\cdot\cdot\mathbf{E})\,dV-\delta W-\Theta_0 k^{-1}\int_V(\tau_0\ddot{\mathbf{B}}+\dot{\mathbf{B}})\cdot\delta\mathbf{B}\,dV+\Theta_0\int_V\vartheta\nabla\cdot\delta\mathbf{B}\,dV-\int_\Phi H\mathbf{n}\cdot\delta\mathbf{B}\,d\Phi=0.\end{gathered} \tag{12}$$

This equation may be used, in particular, to derive the basic energy equation. Suppose that virtual increments on the interval $(t, t+dt)$ coincide with real increments. Then Eq. (12) yields

$$L+A-\dot{K}-\dot{W}+\nu\Theta_0\int_V\vartheta(\dot{\boldsymbol{\gamma}}\cdot\cdot\mathbf{E})\,dV-\frac{1}{\Theta_0 k}\int_V(\tau_0\dot{\mathbf{q}}+\mathbf{q})\cdot\mathbf{q}\,dV+\int_V\vartheta\nabla\cdot\mathbf{q}\,dV-\int_\Phi H\mathbf{n}\cdot\mathbf{q}\,d\Phi=0, \tag{13}$$

where

$$L=\int_V(\mathbf{X}\cdot\dot{\mathbf{u}}+\mathbf{Y}\cdot\dot{\boldsymbol{\omega}})\,dV;\quad A=\int_\Phi(\mathbf{F}\cdot\dot{\mathbf{u}}+\mathbf{P}\cdot\dot{\boldsymbol{\omega}})\,d\Phi;$$

$$K=\frac{1}{2}\int_V[\rho\dot{\mathbf{u}}^2+(\mathbf{I}\cdot\dot{\boldsymbol{\omega}})\cdot\dot{\boldsymbol{\omega}}]\,dV. \tag{14}$$

Using Eqs. (7) and (8) and the third relation in Eq. (2), it is not difficult to show, using the Ostrogradskii–Gauss theorem, that

$$-\frac{1}{\Theta_0 k}\int_V(\tau_0\dot{\mathbf{q}}+\mathbf{q})\cdot\mathbf{q}\,dV+\int_V\vartheta\nabla\cdot\mathbf{q}\,dV-\int_\Phi H\mathbf{n}\cdot\mathbf{q}\,d\Phi=Q+U+\mathrm{X}-\dot{\Gamma}, \tag{15}$$

where

$$Q=k\Theta_0\int_\Phi H\mathbf{n}\cdot\nabla\vartheta\,d\Phi;\quad U=\int_V\vartheta(w+\tau_0\dot{w})\,dV;$$

$$\mathrm{X}=-\Theta_0 k\int_V(\nabla\vartheta)^2\,dV-\tau_0\Theta_0\int_V\ddot{s}\vartheta\,dV;\quad \Gamma=\frac{m\Theta_0^2}{2}\int_V\vartheta^2\,dV. \tag{16}$$

Hence, Eq. (13) may be written in the form

$$L+A+Q+U+X=\dot{W}+\dot{K}+\dot{\Gamma}. \tag{17}$$

Except for the notation used, this equation coincides with the basic energy equation of generalized Cosserat-medium thermoelasticity derived in [7] directly from the complete system of equations in the displacements.

Note that, when $\mathbf{I}=0$, $\mathbf{P}=0$, $\gamma=\varepsilon=\beta=0$, Eq. (12) transforms to the variational equation of generalized symmetric-medium thermoelasticity. Then, in contrast to the equation given in [3], it takes the presence of bulk heat sources into account.

NOTATION

T, shear-stress tensor; **M**, moment-stress tensor; **X**, external-bulk-force vector; **Y**, external-bulk-moment tensor; **u**, displacement vector; ω, small-rotation vector; ρ, density; I, tensor characterizing dynamic properties of medium under rotation; $\mathbf{a}^T$, vector corresponding to tensor T; γ, asymmetric strain tensor; $\varkappa$, flexure—torsion tensor; s, entropy per unit volume; Θ, absolute temperature; Θ_0, initial temperature of medium; ϑ, relative temperature deviation from initial value; **E**, unit tensor; μ, α, λ, ν, γ, ε, β, m, constants characterizing the mechanical and thermophysical properties of the medium; V, volume; Φ, surface; **F**, external-surface-force vector; **P**, external-surface-moment vector; w, bulk-heat-source density; H, temperature distribution at surface; **q**, heat-flux vector; k, thermal conductivity; τ_0, constant characterizing heat-propagation rate; **B**, Biot vector; L, power of external bulk forces and moments; A, power of external surface forces and moments; K, kinetic energy; W, work of deformation; Γ, heat potential; U, power of bulk heat sources; X, dissipation function; Q, thermal power of surface sources.

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